Stochastic vorticity equations: existence and uniqueness results in the flat torus and in the whole plane

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Outline

Introduction: the Navier-Stokes equations

THE NAVIER-STOKES EQUATIONS IN VORTICITY FORM

Let $D \subseteq \mathbb{R}^2$.

NAVIER-STOKES EQUATIONS:

- describe the motion of the fluid.
- $v = (v_1, v_2, 0)$ *vector* field

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = \nu \Delta v - \nabla p + f, \\ \nabla \cdot v = 0, \\ v_{|t=0} = v_0. \\ \text{boundary conditions} \end{cases}$$

Take the *curl* $\downarrow$

$$\xi = curl(v) = \nabla^{\perp} \cdot v$$

VORTICITY EQUATION:

- describes the local rotation of the fluid.
- $\xi = \left(0, 0, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$ *scalar* field.

$$\begin{cases} \frac{\partial \xi}{\partial t} + v \cdot \nabla \xi = \nu \Delta \xi + \text{curl}(f), \\ \nabla \cdot v = 0, \\ \xi = \nabla^{\perp} \cdot v. \\ \text{boundary conditions} \end{cases}$$

TWO DIMENSIONAL PHYSICAL EXAMPLES: atmosphere, ocean.

THE NAVIER-STOKES EQUATIONS IN VORTICITY FORM

In the equation for the vorticity appears the velocity v but...

- v can be recovered from ξ by the Biot-Savart law

$$v(t,x) = (k * \xi(t,\cdot))(x)$$

- k is the Biot-Savart kernel and it has a singularity in zero (in $\mathbb{R}^2$ and $\mathbb{T}^2$):

$$|k| \leq C(|x|^{-1} + 1).$$

Thus...

- Equivalence of formulations for the velocity and vorticity, for enough regular solutions,
- Equation for the vorticity: closed equation for ξ.

SPDEs: martingale measure and functional approaches

Solving a SPDEs: two different approaches

- How do we solve a SPDEs?
- What notion of solution do we consider?

These questions are strictly related to the notion of stochastic integral we consider...

- Stochastic integration theory in Hilbert spaces w.r.t. a cylindrical Wiener process (Da Prato-Zabczyk).
- Stochastic integration theory w.r.t. a martingale measure (Walsh).

In some cases the two notions of solution are equivalent.

The martingale measure approach

- SPDEs with values in $\mathbb{R}$;
- Random field notion of solution: the solution process is evaluated at time t and space x and takes values in $\mathbb{R}$;
- Walsh notion of stochastic integral: is the integral of a real-valued process w.r.t. a martingale measure.

Example: one dimensional stochastic heat equation

$$\frac{\partial X(t,x)}{\partial t} = \frac{\partial^2 X(t,x)}{\partial x^2} + \sigma(X(t,x))\dot{W}(t,x)$$

- Useful if one is interested in the regularity properties of the solution process at fixed points in time and space, e.g. existence of a density for the image law of the solution.
- Literature: heat, wave, Burgers equations.
- Never used for the Navier-Stokes equations.

The functional approach

- stochastic evolution equations with values in a Hilbert (or Banach) space H;
- the solution process is evaluated in time t and takes values in H;
- Da Prato-Zabczyk notion of stochastic integral: is the integral w.r.t. a cylindrical Wiener process.

Example: one dimensional stochastic heat equation

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + G(X(t))\mathrm{d}W(t)$$

- Is the approach generally used for the study of the Navier-Stokes equations.

Analysis on the flat torus $\mathbb{T}^2$

Existing literature and new results

- There exists a fairly well-developed literature for what concerns the stochastic Navier-Stokes equations on the flat torus.
- The existence and uniqueness of the solution process is a well understood problem. The results available in literature are based on the functional (Da Prato-Zabczyk) approach.

What did we prove?

- Existence, uniqueness and space-time continuity of the solution process in the martingale measure approach.

Why did we use this approach?

- We are interested in the regularity of the solution process, evaluated at fixed points in time and space, in the Malliavin sense.

Regularity in Malliavin sense: General Ideas

AIM: prove the existence of a density for the image law of the solution process $\xi(t,x)$ at fixed $(t,x) \in [0,T] \times D$.

$$\begin{aligned} \xi(t,x) :&(\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) \\ &\mu(A) = \mathbb{P}(\xi(t,x) \in A), \qquad A \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

We ask whether μ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$, i.e. there exists a density ρ s.t. $\mu(A) = \int_A \rho(x)\, dx$.

TOOLS: Malliavin calculus.

EXISTING LITERATURE (fluid dynamics): one dimensional stochastic Burgers equation.

Vorticity equation on $\mathbb{T}^2$

For $(t, x) \in [0, T] \times D$, (with $D = [0, 2\pi]^2$)

$$\begin{cases} \frac{\partial \xi}{\partial t}(t,x) - \Delta\xi(t,x) + v(t,x)\cdot\nabla\xi(t,x) = \sigma(\xi(t,x))w(\mathrm{d}x,\mathrm{d}t) \\ \nabla\cdot v(t,x) = 0 \\ \xi(t,x) = \nabla^{\perp}\cdot v(t,x) \\ \xi(0,x) = \xi_0(x) \\ \text{periodic boundary conditions} \end{cases} \tag{1}$$

$w(\mathrm{d}x, \mathrm{d}t)$ is the formal notion for a Gaussian noise white in time coloured in space.

(H1): σ satisfies a linear growth condition and it is globally Lipschitz

(H2): σ is bounded.

Walsh notion of solution

Let $g(t,x,y)$ be the fundamental solution to the heat equation on the flat torus. A random field (r.f.) $\xi = \{\xi(t,x), t \in [0,T] \times D\}$ is a solution to equation (1) if it satisfies the evolution equation

$$\xi(t,x) = \int_D g(t,x,y)\xi_0(y)\,\mathrm{d}y + \int_0^t \int_D \nabla_y g(t-s,x,y) \cdot v(s,y)\xi(s,y)\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_D g(t-s,x,y)\sigma(\xi(s,y))\,w(\mathrm{d}y,\mathrm{d}s) \qquad (2)$$

with $v = k * \xi$.

- Stochastic integral understood in the Walsh sense w.r.t. a worthy martingale measure.
- "Simple case" example: Itô integral, stochastic integral w.r.t. a space-time white noise.

Constructing a worthy martingale measure

Intuitively: the heat kernel becomes less smooth as the dimension increases: for the well posedness of the stochastic integral a weight in space is needed.

Formally: introduce the worthy martingale measure starting from an isonormal Gaussian process $W = \{W(h), h \in \mathcal{H}_T\}$.

- Let

$$Q = (-\Delta)^{-b}, \text{ for some } b > 0. \tag{3}$$

- Let $\mathcal{H}_T = L^2(0, T; L^2_Q(D))$. endowed with the scalar product

$$\langle f, g \rangle_{\mathcal{H}_T} = \int_0^T \langle f(s), g(s) \rangle_{L^2_Q(D)} \, \mathrm{d}s = \int_0^T \langle Qf(s), g(s) \rangle_{L^2(D)} \, \mathrm{d}s.$$

- The covariance operator Q provides the regularization in space.
- From W we construct the worthy martingale measure w.
- Characterization of the class of predictable processes.

Existence and uniqueness of a r.f. solution

Theorem

Let $b > 0$ in (3) and $p > 2$. Let us assume that Hypothesis (H1)-(H2) hold. If $\xi_0 \in L^p(D)$, then there exists a unique solution to equation (2) which is continuous with values in $L^p(D)$.

KEY POINTS OF THE PROOF

- The non linear term is non Lipschitz: we introduce a truncation factor $\Theta_N(\|\xi\|_{L^p(D)})$, $p > 2$, where $\Theta_N : [0, +\infty) \to [0, 1]$ a C^1 function s.t. $|\Theta_N'(s)| \leq 2$ for any $s \geq 0$ and $\Theta_N(s) = \begin{cases} 1 & \text{if } 0 \leq s < N \\ 0 & \text{if } s \geq N+1. \end{cases}$
- Exploiting the Biot-Savart law we have a control also on v.
- $\Theta_N(|\xi(t,x)|)$ not suitable.
- We use a stopping time argument: we prove the existence of local solution and then we pass to the limit.

Existence of a space-time continuous modification

Theorem

Let $b > 0$ in (3) and $p > 2$. Let us assume that Hypothesis (H1)-(H2) hold. If $\xi_0 \in C(D)$ the solution admits a modification which is a space-time continuous process.

KEY POINTS OF THE PROOF AND A CONSIDERATION

- The regularizing effect of the heat kernel and its gradient.
- With the martingale measure approach we require the minimal hypothesis on the covariance Q.

Existence of a density

We require more regularity on the covariance function σ.

(H3): σ is of class C^1 on $\mathbb{R}$ and has first derivative bounded;

(H4): there exists $\sigma_0 > 0$ such that $|\sigma(x)| \geq \sigma_0$ for all $x \in \mathbb{R}$.

Theorem

Let $b > 1$ in (3) and assume that hypothesis (H1)-(H4) hold. If $\xi_0 \in C(D)$, then for every $t \in [0, T]$ and $x \in D$ the image law of the random variable $\xi(t, x)$ is absolutely continuous w.r.t. to the Lebesgue measure on $\mathbb{R}$.

The proof is based on a Malliavin analysis of the solution process at fixed points in time and space.

Analysis on the whole plane $\mathbb{R}^2$

Main differences from the $\mathbb{T}^2$ case

AIM: Investigate the existence and uniqueness of the solution.

MAIN SOURCE OF DIFFICULTY: the domain is not compact.

- The technique used for $\mathbb{T}^2$ can not be readapted in the case of the whole space.
- The singularity of the Biot-Savart kernel prevents us to obtain the needed estimates that allows us to the treat the vorticity equation as a closed equation for ξ.
- On a non compact domain this closed form is difficult to handle; indeed, $k \notin L^p(\mathbb{R}^2)$ for any $1 \leq p \leq \infty$.

  $\longrightarrow$ we take into account the equations for the velocity.
- When v exists and has a suitable regularity, we can handle the equations for the vorticity.

The equations in $\mathbb{R}^2$

- In this case we work in the functional approach.
- We consider both the equations for the velocity and the vorticity.

Navier-Stokes equations

$$\begin{cases} \partial_t v + [-\Delta v + (v \cdot \nabla) v + \nabla p] \, dt = G(v) \, \partial_t W \\ \nabla \cdot v = 0, \\ v(0, x) = v_0(x). \end{cases}$$

Vorticity equations

$$\begin{cases} \partial_t \xi + [-\Delta \xi + v \cdot \nabla \xi] \, dt = \operatorname{curl}(G(v) \, \partial_t W) \\ \nabla \cdot v = 0 \\ \xi = \nabla^{\perp} \cdot v, \\ \xi(0, x) = \xi_0(x). \end{cases}$$

Hypothesis on the noise

We consider $q > 2$ and assume that there exists $g \in (0,1)$ such that

(IG1): The mapping $G : \mathbb{L}^2 \to L_{\rm HS}(\mathcal{H}; H^{1-g,2})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|G(v)\|_{L_{\rm HS}(\mathcal{H};H^{1-g,2})} =: C_{g,2} < \infty,$$

(IG2):The mapping $G : \mathbb{L}^2 \to R(\mathcal{H}; H^{1-g,q})$ is well defined and

$$\sup_{v\in\mathbb{L}^2} \|G(v)\|_{R(\mathcal{H};H^{1-g,q})} =: C_{g,q} < \infty.$$

(IG3): If assumption **(IG1)** holds, then for any $\varphi \in H^{1-g,2}$ and any $v \in \mathbb{L}^2$ the mapping $v \to G(v)^*\varphi \in \mathcal{H}$ is continuous when in $\mathbb{L}^2$ we consider the Fréchet topology inherited from the space $\mathbb{L}^2_{loc}$ or the weak topology of $\mathbb{L}^2$.

(IG4): If assumption **(IG1)** holds, then G ia s Lipschitz continuous function when we consider a weak norm, i.e.

$$\exists\ L_g > 0 : \|G(v_1) - G(v_2)\|_{L_{HS}(\mathcal{H};\mathbb{L}^2)} \leq L_g\|v_1 - v_2\|_{\mathbb{L}^2},\ \forall v_1, v_2 \in \mathbb{L}^2$$

The equations for the velocity

- Existence and uniqueness of L^2-solutions of the stochastic Navier-Stokes equations in unbounded domains is not trivial: the direct application of the compactness method, which is central in the proof, fails.
- Main source of difficulty: the embedding $H^{1,2} \subset L^2$, unlike in the bounded space, is not compact.
- We prove the required regularity on v starting from the work

Z. Brzeźniak and E. Motyl, *Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains*, J. Differential Equations, 254 (2013), pp. 1627–1685.

The noise driving the vorticity equation

- The noise driving the equation for the velocity is rather good!
- The noise driving the equation for the vorticity is not: taking the *curl* we loose one order of differentiability in space.
- The noise is not regular enough to allow to use Itô formula in the space of finite energy velocity vectors, which is the basic space in which one looks for existence of solutions.
- We introduce an approximation system by regularizing the covariance of the noise. We follow the ideas of the work

Z. Brzeźniak and B. Ferrario, *A note on stochastic Navier-Stokes equations with not regular multiplicative noise*, Stoch. Partial Differ. Equ. Anal. Comput., 5 (2017), pp. 53–80.

Martingale solution

A martingale solution to the vorticity equation is a triple consisting of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, an $\{\mathcal{F}_t\}$-adapted cylindrical Wiener process W on $\mathcal{H}$ and an $\{\mathcal{F}_t\}$-adapted measurable process ξ such that $\xi : [0, T] \times \Omega \to L^2$ with $\mathbb{P}$-a.a. paths

$$\xi(\cdot, \omega) \in C([0, T] \, ; L^2),$$

and such that for all $z \in C^\infty_{\mathrm{sol}}$ and $t \in [0, T]$

$$\langle \xi(t), z \rangle = \langle \xi_0, z \rangle + \int_0^t \langle \xi(s), \Delta z \rangle \, \mathrm{d}s + \int_0^t \langle v(s)\xi(s), \nabla z \rangle \, \mathrm{d}s + \langle \int_0^t \tilde{G}(v(s)) \, \mathrm{d}W(s), z \rangle$$

$\mathbb{P}$-a.s., where v is the solution to the equation for the velocity.

- Stochastic integral understood in the Da Prato-Zabczyk sense.

Main result

Theorem

1. *Let $q = 4$ and assume conditions* **(IG1)**, **(IG2)**, **(IG3)**. *Let $\xi_0 \in L^2$ and $v_0 \in \mathbb{L}^2$. Then there exists a martingale solution $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \tilde{W}, \tilde{\xi})$ to the vorticity equation. In addition $\tilde{\xi} \in L^4(0, T; L^4)$ $\mathbb{P}$-a.s..*

2. *If, in addition, we assume that condition* **(IG2)** *holds also for a $q > 2$, and $\xi_0 \in L^2 \cap L^q$, $v_0 \in \mathbb{L}^2 \cap \mathbb{L}^q$, then also $\tilde{\xi} \in L^\infty(0, T; L^q)$ $\mathbb{P}$-a.s..*

3. *Under the same assumptions of Point (1) and* **(IG5)**, *there exists a unique strong solution to the vorticity equation.*

Key points of the proof

- Step 1: APPROXIMATION. We introduce an approximation system by regularizing the covariance of the noise (Hille-Yosida approximations).
  We construct a sequence of approximating processes $\{\xi_n\}_n$.
- Step 2: UNIFORM ESTIMATES IN n. We work pathwise with two auxiliary processes β_n and ζ_n with $\xi_n = \beta_n + \zeta_n$.
- Step 3: TIGHTNESS OF THE LAW OF THE APPROXIMATIONS $\{\xi_n\}_n$.
- Step 4: PASSAGE TO THE LIMIT.
- Step 5: PATHWISE UNIQUENESS. Proved for v by a classical argument.

SOME REMARKS.

- The compactness method is based on a modification of the classical Dubinsky compactness theorem that allows to work in unbounded domains.
- We work locally and with weak topologies.

Thank you for your attention!