Application of optimal transport to a congestion traffic model

SCUOLA DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE
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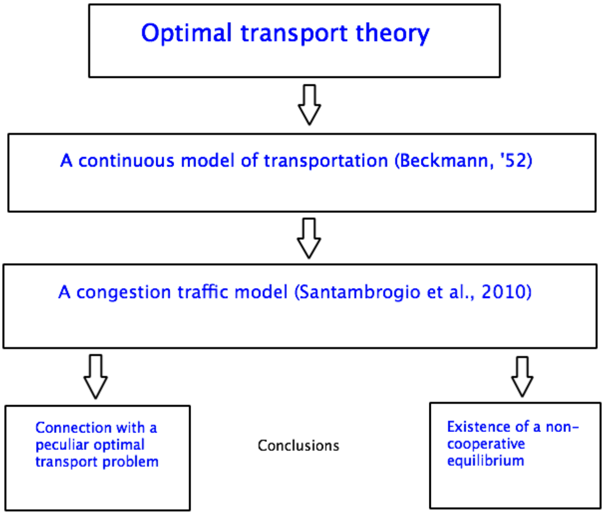
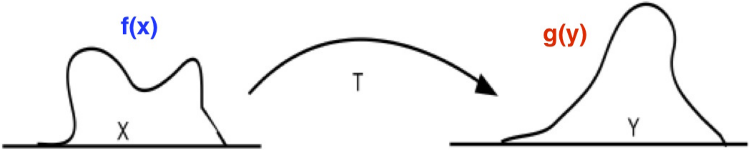
Plan of the work



The original optimal transport problem



Goal: minimize the quantity

$$M(T) := \int_{\mathbb{R}^d} |T(x) - x| \, f(x) \, \mathrm{d}x, \qquad (1)$$

among all the maps satisfying this condition

$$\int_A g(y) \, \mathrm{d}y = \int_{T^{-1}(A)} f(x) \, \mathrm{d}x \quad \forall A \in \mathscr{B}(\mathbb{R}^d). \qquad (2)$$

The Monge problem

Given

- X and Y two metric spaces;
- two probability measures $\mu \in \mathscr{M}^1_+(X)$ and $\nu \in \mathscr{M}^1_+(Y)$;
- a Borel cost function $c : X \times Y \to \mathbb{R}_+$;

we look for

$$\inf \left\{ \int_X c(x, T(x))\, \mathrm{d}\mu(x) \ : \ T : X \to Y \text{ Borel}, \ T_\# \mu = \nu \right\}, \tag{MP}$$

where $T_\# \mu$ is the measure on Y characterized by

$$(T_\# \mu)(A) = \mu(T^{-1}(A)) \ \forall A \in \mathscr{B}(Y). \tag{3}$$

The Monge problem

Difficulties of the Monge formulation

- a borel map T, satisfying the mass constraint, may not exist;
- the mass constraint is not closed with respect to the weak convergence.

Indeed, for example, consider

1. $\mu = \delta_{x_0}$ and $\nu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$;
2. $\mu = f(x)\,\mathrm{d}x$ and $\nu = g(y)\,\mathrm{d}y$, the constraint is

$$g(T(x))\det(DT(x)) = f(x).$$

The Kantorovich problem

Given

- two probability measures $\mu \in \mathscr{M}_+^1(X)$ and $\nu \in \mathscr{M}_+^1(Y)$;
- a Borel cost function $c : X \times Y \to \mathbb{R}_+$;

Set of transport plans

$$\Pi(\mu,\nu) := \{\pi \in \mathscr{M}_+^1(X \times Y) \ : \pi(A \times Y) = \mu(A),\ \pi(X \times B) = \nu(B) \\ \forall A \in \mathscr{B}(X), B \in \mathscr{B}(Y)\}.$$

We look for

$$\inf \left\{ \int_{X \times Y} c(x,y)\, \mathrm{d}\pi(x,y) \ : \ \pi \in \Pi(\mu,\nu) \right\}. \tag{KP}$$

The Kantorovich problem

Advantages of the Kantorovich formulation

- $\Pi(\mu, \nu)$ is always not empty;
- it is a linear problem under linear constraint.

Indeed notice that

$$\pi = \mu \otimes \nu \in \Pi(\mu, \nu).$$

Moreover, thanks to the linearity of the problem, we can now prove an existence result.

Existence result

Theorem

Let X and Y be compact metric spaces, $\mu \in \mathscr{M}^1_+(X)$, $\nu \in \mathscr{M}^1_+(Y)$ and $c : X \times Y \to \mathbb{R}_+$ a continuous function.
Then the Kantorovich problem admits a solution.

In order to prove it, we just need to apply the direct method of calculus of variations.

Connection between transport maps and transport plans

Given a transport map T, we can define the transport plan $\pi = (id \times T)_{\#}\mu$, where $id \times T : X \to X \times Y$ is the obvious function. Obviously

$$\int_X c(x, T(x))\,\mathrm{d}\mu(x) = \int_{X\times Y} c(x,y)\,\mathrm{d}\pi,$$

thus

$$\inf_{(\mathrm{MP})} \geq \inf_{(\mathrm{KP})}.$$

But what about the converse?

Theorem (A. Pratelli)

If μ has no atom and $c : X \times Y \to [0,\infty)$ is continuous, then

$$\inf_{(\mathrm{MP})} = \min_{(\mathrm{KP})},$$

where the infimum in the Kantorovich's formulation is a minimum.

Definition (Beckmann's problem)

Let $q(x, y)$ and $H(x, y, u)$ be smooth enough fields defined on the domain $\Omega \subset \mathbb{R}^2$ and let $g(x, y)$ be a continuous function defined on $\partial\Omega$. Supposing

- $H(x, y, u) > 0$;
- $\partial_u H(x, y, u) > 0$;
- $\partial_u^2 H > 0$;

we want to minimize

$$\int_\Omega H(x, y, |\phi|) \, \mathrm{d}x \, \mathrm{d}y$$

over the class of vector fields ϕ, subject to the two following constraints:

- $\nabla \cdot \phi(x, y) = q(x, y)$ in Ω;
- $\phi(x, y) \cdot n = g(x, y)$ on $\partial\Omega$.

Connection between the Beckmann's problem and optimal transport theory

Let us now slightly particularize the previous model in order to better see the connections with the optimal transport theory. We consider

- μ, $\nu \in \mathscr{M}^1_+(\bar{\Omega})$, that represent, after normalization, the distribution of consumption and of production respectively;
- the domain Ω is isolated, i.e. $\phi \cdot n = 0 \quad \text{on } \partial\Omega$;
- $H(x, y, u) = u$.

Notice that now, the local measure of excess demand is $\mu - \nu$.

Beckmann's problem:

$$(\text{BP}) \;\; \inf\{|\phi|(\Omega) \; : \; \phi \in \mathscr{M}^d_{\text{div}}, \nabla \cdot \phi = \mu - \nu\}.$$

Connection between the Beckmann's problem and optimal transport theory

In the following theorem we will see how this problem is related to the classical Monge-Kantorovich problem with cost $c(x, y) = |x - y|$, i.e.:

$$\text{(KP)} \quad \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| \, \mathrm{d}\gamma(x, y). \tag{4}$$

Theorem

The value of the minimal flow problem (BP) coincides with (KP). Moreover, if γ is an optimal transport plan for (KP), then the vector-valued measure ϕ_γ defined by

$$\langle \phi_\gamma, X \rangle := \int_{\bar{\Omega} \times \bar{\Omega}} \left(\int_0^1 X(x + t(y - x)) \cdot (y - x) \, \mathrm{d}t \right) \mathrm{d}\gamma(x, y), \quad \forall X \in C(\bar{\Omega}, \mathbb{R}^d)$$

is a solution of Beckmann's problem (BP).

Congestion traffic model

In orther to model congestion we need the following tools:

- $C := W^{1,\infty}([0,1],\bar{\Omega})$, viewed as a subspace of $C^0([0,1],\mathbb{R}^2)$, i.e., equipped with the uniform topology;
- $C^{x,y} := \{\sigma \in C : \sigma(0) = x, \sigma(1) = y\}$ (x, y in $\bar{\Omega}$);

Weighted length

For $\phi \in C^0(\bar{\Omega},\mathbb{R})$ and $\sigma \in C$ we define

$$L_\phi(\sigma) := \int_0^1 \phi(\sigma(t))\,|\sigma'(t)|\,\mathrm{d}t. \tag{5}$$

Congestion traffic model

The first step is to have a tool in order to quantify how much busy a path is. Thus we explicitly introduce probabilities over $C^{x,y}$.

Definition (Transportation strategy)

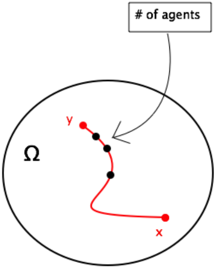
A transportation strategy consist of a pair (γ, p) with:

- $\gamma \in \Pi(\mu_0, \mu_1)$;
- $p = (p^{x,y})_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} \subset \mathscr{M}^1_+(C)$ such that $p^{x,y}(C^{x,y} = 1)$ for γ-a.e. $(x, y) \in \bar{\Omega} \times \bar{\Omega}$.

Congestion traffic model

Definition: Overall probability over paths $Q_{\gamma,p} \in \mathscr{M}^1_+(C)$

$$\int_C F(\sigma)\,\mathrm{d}Q_{\gamma,p}(\sigma) = \int_{\bar\Omega\times\bar\Omega}\Big(\int_{C^{x,y}} F(\sigma)\,\mathrm{d}p^{x,y}\Big)\,\mathrm{d}\gamma(x,y) \text{ for all } F \in C^0(C,\mathbb{R}). \quad (6)$$

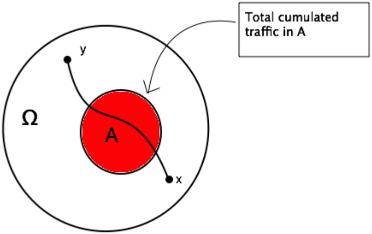


Congestion traffic model

Definition: Traffic intensity $I_{\gamma,p} \in \mathscr{M}_+(\bar{\Omega})$

Setting $Q := Q_{\gamma,p} \in \mathscr{M}^1_+(C)$, then we define $I_{\gamma,p} = i_Q \in \mathscr{M}_+(\bar{\Omega})$ as:

$$\int_{\bar{\Omega}} \phi(x)\,\mathrm{d}i_Q(x) = \int_C L_\phi(\sigma)\,\mathrm{d}Q(\sigma) \quad \forall \phi \in C^0(\bar{\Omega}, \mathbb{R}_+). \tag{7}$$



The new cost function

$$c_{\gamma,p}(x,y) = \int_{C_{x,y}} L_{G_{I_{\gamma,p}}}(\sigma)\,\mathrm{d}p^{x,y}(\sigma), \tag{8}$$

where

$$G_i(x) = g(\frac{di}{d\mathcal{L}^2}(x)) \tag{9}$$

s.t.

- $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function;
- $\lim_{z\to+\infty} g(z) = +\infty$;
- the function H defined by $H(z) = zg(z)$ for all $z \in \mathbb{R}_+$ is convex.

Congestion traffic model

Supposing:

- $i \ll \mathcal{L}^2$;
- $H(z) := zg(z)$ convex;

Definition (Optimal transportation strategy)

A transportation strategy (γ, p) is said to be optimal if the overall probability $Q_{\gamma,p}$ solves the minimization problem

$$\inf_{Q \in Q(\mu_0,\mu_1)} \int_\Omega H(i(x))\,\mathrm{d}x \tag{10}$$

Existence result

The ingredients of our recipe will be:

- there exist $q > 1$ and positive constants a and b such that
$$az^q \leq H(z) \leq b(z^q + 1) \quad \forall z \in \mathbb{R}_+; \tag{11}$$
- H is differentiable on $\mathbb{R}_+$, and there exists a positive constant c such that $0 \leq H'(z) \leq c(z^{q-1} + 1)$ for all $z \in \mathbb{R}_+$;
- the set
$$\mathcal{Q}^q(\mu_0, \mu_1) := \{Q \in \mathcal{Q}(\mu_0, \mu_1) : i_Q \ll \mathcal{L}^2 \text{ and } \frac{di_Q}{d\mathcal{L}^2} \in L^q\} \tag{12}$$
is non-empty.

Theorem

The minimization problem

$$\inf_{Q \in \mathcal{Q}^q(\mu_0, \mu_1)} \int_\Omega H(i_Q(x))\, \mathrm{d}x \tag{13}$$

admits a solution.

Characterization of the minimizers

The minimizer of Problem 13 satisfies the following variational problem.

Theorem

$\bar{Q} \in \mathcal{Q}^q(\mu_0, \mu_1)$ *solves the optimization Problem (13) if and only if*

$$\int_\Omega \bar{\xi} i_{\bar{Q}} = \inf\{\int_\Omega \bar{\xi} i_Q : Q \in \mathcal{Q}^q(\mu_0, \mu_1)\} \tag{14}$$

where

$$\bar{\xi} := H'(i_{\bar{Q}}) \in L^{q*}.$$

Congestion traffic model

Non-cooperative transportation cost

$$c_{\bar{\xi}}(x,y) := \inf_{\sigma \in C^{x,y}} L_{\bar{\xi}}(\sigma). \tag{15}$$

Then $\bar{\lambda} \in \Pi(\mu_0, \mu_1)$, solution the optimization Problem (13), solves also

Non-cooperative Kantorovich problem

$$\int_{\bar{\Omega}\times\bar{\Omega}} c_{\bar{\xi}}(x,y)\,\mathrm{d}\bar{\gamma}(x,y) = \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\bar{\Omega}\times\bar{\Omega}} c_{\bar{\xi}}(x,y)\,\mathrm{d}\gamma(x,y). \tag{16}$$

Congestion traffic model

Wardrop equilibrium condition

$$L_{\bar{\xi}}(\sigma) = c_{\bar{\xi}}(\sigma(0), \sigma(1)) \quad \text{for } \bar{Q}\text{-a.e. } \sigma, \tag{17}$$

or, in a equivalent way, for $\bar{\gamma}$-a.e. (x, y) one has

$$L_{\bar{\xi}}(\sigma) = c_{\bar{\xi}}(x, y) \quad \text{for } \bar{p}^{x,y}\text{-a.e. } \sigma. \tag{18}$$

Complete characterization of the minimizer

Theorem

Let us assume that $q < 2$ and that H is strictly convex.
A transportation strategy $(\bar{\gamma}, \bar{p})$ is optimal, if and only if, setting $\bar{Q} := Q_{\bar{\gamma},\bar{p}}$ and $\bar{\xi} := H'(i_{\bar{Q}})$, one has that

- *$\bar{\gamma}$ solves the Monge–Kantorovich problem*

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\bar{\Omega}\times\bar{\Omega}} \bar{c}_{\bar{\xi}}(x,y)\,\mathrm{d}\gamma(x,y); \tag{19}$$

- *for $\bar{Q}$-a.e. $\sigma \in C$, one has*

$$L_{\bar{\xi}}(\sigma) = c_{\bar{\xi}}(\sigma(0), \sigma(1)). \tag{20}$$

Difficulties and how to deal with them

Warning

- the definition of the weighted Length L_ϕ, when the weight ϕ is only in L^{q*};
- why $q < 2$?

New ideas and further research

What makes traffic situations very chaotic are random phenomena.

New questions

- How to take care about random effects in traffic models?
- If I impose a random constraint on the family $p = (p^{x,y})_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} \subset \mathscr{M}^1_+(C)$, will an efficient equilibrium solution still exist?
- How can I compute it?

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Dual formulation

Definition

Energy functional

$$J(\xi) := \int_\Omega H^*(x, \xi(x))dx - \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x, y)d\gamma(x, y). \qquad (21)$$

The dual problem of $(\mathcal{P})$:

$$(\mathcal{P}^*) \quad \sup\{-J(\xi) : \xi \in L^{q^*}, \xi \geq \xi_0\}. \qquad (22)$$

The discrete energy functional

The functional to be minimized is the following

The discrete functional

$$J^h(\xi) = h^2 \sum_{i,j} H^*(x_{i,j}, \xi_{i,j}) - \sum_{\alpha,\beta} c_\xi^h(S_\alpha, T_\beta)\gamma_{\alpha,\beta}, \tag{23}$$

where the weights $\gamma_{\alpha,\beta}$ represent the coupling on the set of pairs sources S_α - targets T_β and $\sum_{\alpha,\beta} \gamma_{\alpha,\beta} = 1$.

The minimization algorithm

We use the subgradient method, which corresponds (taking for simplicity $\xi_0 = g(.,0) = 0$) to the following iterative scheme

subgradient method

$$\xi^{(1)} = 1; \quad \xi^{(k+1)} = \max\{0, \xi^{(k)} - \rho_k w^{(k)}\} \tag{24}$$

$$\text{where} \quad (w^{(k)})_{i,j} = f'(\xi^{(k)}_{i,j}) + (v^{(k)})_{i,j} \in \vartheta J(\xi^{(k)}),$$

$v^{(k)} \in \vartheta K(\xi^{(k)})$ is a vector in the subdifferential of K at the previous point $\xi^{(k)}$ and (ρ_k) is a suitable sequence of steps.

Traffic equilibria on a finite network

The main data of the model are

- a finite oriented connected graph $G = (N, E)$ modeling the network;
- a travel times functions $g_e : \theta \in \mathbb{R}_+ \mapsto g_e(\theta)$;
- a transport plan on pairs of nodes $(\gamma_{x,y})_{(x,y)\in N^2}$;
- $C := \cup_{(x,y)\in N^2} C_{x,y}$ is the set of all simple paths;

The unknown of the problem is the flow configuration. The edge flows are denoted by $i = (i_e)_{e\in E}$ and the path flows are denoted by $q = (q_\omega)_{\omega\in C}$. The total cost is

$$L_i(\omega) = \sum_{e\in\omega} g_e(i_e). \tag{25}$$

Definition

A Wardrop equlibrium is a flow configuration $q = (q_\omega)_{\omega \in C}$ satisfying $q_\omega \geq 0$ and the mass conservation constraints, such that, when we compute the values i_e, for every $(x, y) \in N^2$ and every $\omega \in C_{x,y}$ with $q_\omega > 0$ we have

$$L_i(\omega) = \min_{\omega' \in C_{x,y}} L_i(\omega'). \tag{26}$$

Theorem

The flow configuration $q = (q_\omega)_{\omega \in C}$ is a Wardrop equilibrium if and only if it solves the convex minimization problem

$$min \left\{ \sum_{e \in E} H_e(i_e) : q \geq 0 \, satisfies \; the \; mass \; conservation \; constraint \right\} \tag{27}$$

where, for each e, we define H_e to be an antiderivative of $g_e : H_e' = g_e$.